Valuation of Variable Annuity Guarantees

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Background

- The term variable annuity is used to refer to a wide range of life insurance products, whose benefits can be protected against investment and mortality risks by selecting one or more guarantees out of a broad set of possible arrangements.
- Variable annuities were introduced first in USA, in 1950s
- In 1990s, insurers included certain guarantees in such policies, guaranteed minimum death benefits (GMDB), guaranteed minimum living benefits (GMLB).

Background

- VAs were also successfully introduced in Asia market, e.g. in Japan, the volume of such contracts has grown to more than USD 100 bn.
- VAs become popular in Europe
- However, due to the complexity of such contracts (their valuation and hedging), some countries hesitate to offer VAs.

Introduction

- Option's payoff, e.g. European call: $(S_T - K)_+$
- Contingent option's payoff: $(S_\tau - K)_+$, where τ is a random variable, independent of S_t.

Literature review

- Milevsky, M. A., Posner, S. E., 2001. The titanic option: valuation of the guaranteed minimum death benefit in variable annuities and mutual funds, The Journal of Risk and Insurance, 68 (1), 93-128. ‘
- Ulm, E. R., 2006. The effect of the real option to transfer on the value of guaranteed minimum death benefits, The Journal of Risk and Insurance, 73 (1), 43-69.
- Ulm, E. R., 2008. Analytic solution for return of premium and rollup guaranteed minimum death benefit options under some simple mortality laws, ASTIN Bulletin, 38 (2), 543-563.

Distribution of τ

- Any distribution on $(0, \infty)$ can be approximated by a linear combination of exponential distributions
- $$f_\tau(t) = \sum_{i=1}^{n} A_i \lambda_i e^{-\lambda_i t}, \qquad t > 0,$$

Distribution of τ

$$
\begin{aligned}
& \mathsf{E}[e^{-\delta\tau}\Pi(S(\tau))] \\
= & \int_0^\infty e^{-\delta t}\mathsf{E}[\Pi(S(t))]f_\tau(t)dt \\
= & \int_0^\infty e^{-\delta t}\mathsf{E}[\Pi(S(t))]\left[\sum_{i=1}^n A_i f_i(t)\right]dt \\
= & \sum_{i=1}^n A_i \int_0^\infty e^{-\delta t}\mathsf{E}[\Pi(S(t))]f_i(t)dt.
\end{aligned}
$$

Brownian motion (Wiener process)

- $X(t) = \mu t + \sigma W(t)$
- $\{W(t)\}$: standard Wiener process
- notation: $D = \frac{\sigma^2}{2}$
- running maximum: $M(t) = \max_{0 \le s \le t} X(s)$

Three probability density functions:

$f_{X(t)}(x)$: pdf of $X(t)$

$f_{M(t)}(m)$: pdf of $M(t)$

$f_{X(t),M(t)}(x, m)$: joint pdf of $X(t)$ and $M(t)$

Three probability density functions:

$$
\begin{aligned}
f_{X(t)}(x) &= \frac{1}{\sqrt{2\pi t}\sigma} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}, \quad -\infty < x < \infty \\
f_{M(t)}(m) &= \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(m-\mu t)^2}{2\sigma^2 t}} - \frac{2\mu}{\sigma^2} e^{\frac{2\mu m}{\sigma^2}} \Phi\left(\frac{-m-\mu t}{\sigma\sqrt{t}}\right) \\
&\quad + \frac{1}{\sigma\sqrt{2\pi t}} e^{\frac{2\mu m}{\sigma^2} - \frac{(m+\mu t)^2}{2\sigma^2 t}}, \quad m \geq 0
\end{aligned}
$$

$$
\begin{aligned}
f_{X(t),M(t)}(x,m) &= \frac{2(2m-x)}{\sigma^3\sqrt{2\pi t^3}} e^{(\mu x - \frac{1}{2}\mu^2 t - \frac{(2m-x)^2}{2t})\sigma^{-2}}, \\
&\qquad -\infty < x \leq m, m \geq 0
\end{aligned}
$$

2 Exponential stopping of Brownian motion

- τ: exponential random variable
  independent of $\{X(t)\}$
  $f_\tau(t) = \lambda e^{-\lambda t}, \quad t > 0$
- We are interested in $X(\tau)$, $M(\tau)$, ...
- δ: force of interest used for discounting

Three discounted density functions

- $f^{\delta}_{X(\tau)}(x) = \int_0^\infty e^{-\delta t} f_{X(t)}(x) f_\tau(t) dt$

- $f^{\delta}_{M(\tau)}(m) = \int_0^\infty e^{-\delta t} f_{M(t)}(m) f_\tau(t) dt$

- $f^{\delta}_{X(\tau),M(\tau)}(x, m) = \int_0^\infty e^{-\delta t} f_{X(t),M(t)}(x, m) f_\tau(t) dt$

Theorem

$\alpha < 0$ and $\beta > 0$ solutions of the quadratic equation $D\xi^2 + \mu\xi - (\lambda + \delta) = 0$

- 1). $f^{\delta}_{X(\tau),M(\tau)}(x, m) = \frac{\lambda}{D} e^{-\alpha x - (\beta - \alpha)m} = \frac{\lambda}{D} e^{\alpha(m-x) - \beta m},$
  $-\infty < x \le m, \ \ m \ge 0$

- 2). $f^{\delta}_{M(\tau)}(m) = \frac{\lambda}{\lambda+\delta} \beta e^{-\beta m}, \qquad m \ge 0$

  Kyprianou (2006, Equ.8.2)

- 3). $f^{\delta}_{X(\tau)}(x) = \begin{cases} \frac{\lambda}{D(\beta-\alpha)} e^{-\alpha x}, & \text{if } \ x < 0, \\ \frac{\lambda}{D(\beta-\alpha)} e^{-\beta x}, & \text{if } \ x > 0. \end{cases}$

  Albrecher, Cheung, Thonhauser (2010, Ex.4.1)

Comparison of discounted density functions with probability density functions

$$f^{\delta}_{X(\tau),M(\tau)}(x,m) = \frac{\lambda}{D} e^{\alpha(m-x)-\beta m}$$

versus

$$f_{X(t),M(t)}(x,m) \quad = \quad \frac{2(2m-x)}{\sigma^3\sqrt{2\pi t^3}} e^{(\mu x - \frac{1}{2}\mu^2 t - \frac{(2m-x)^2}{2t})\sigma^{-2}}$$

Note that 2) and 3) follow from 1):

$$
\begin{aligned}
f^{\delta}_{M(\tau)}(m) &= \int_{-\infty}^{m} f^{\delta}_{X(\tau),M(\tau)}(x,m)dx \\
&= \int_{-\infty}^{m} \frac{\lambda}{D} e^{\alpha(m-x)-\beta m} dx = \frac{\lambda}{\lambda+\delta}\beta e^{-\beta m} \\
f^{\delta}_{X(\tau)}(x) &= \int_{\max(x,0)}^{\infty} f^{\delta}_{X(\tau),M(\tau)}(x,m)dm \\
&= \kappa\, e^{-\alpha x-(\beta-\alpha)\max(x,0)} \\
&= \begin{cases} \kappa e^{-\alpha x}, & \text{if } \; x<0, \\ \kappa e^{-\beta x}, & \text{if } \; x>0. \end{cases}
\end{aligned}
$$

with $\kappa = \frac{\lambda}{D(\beta-\alpha)}$

Proof of 1):

- Does not use $f_{X(t),M(t)}(x, m)$
- Does not use the reflection principle
- Idea: For an arbitrary bounded function $\pi(u, m)$, consider

$$V(u, m) = E[e^{-\delta\tau}\pi(u + X(\tau), \max(u + M(\tau), m))]$$

In particular, determine $V(0, 0)$

$$D\frac{\partial^2 V}{\partial u^2} + \mu\frac{\partial V}{\partial u} - (\lambda+\delta)V + \lambda\pi = 0$$

Note that $\alpha < 0$ and $\beta > 0$ are the roots of the characteristic equation.

$$\frac{\partial V}{\partial m}(u,m)|_{u=m} = 0, \quad \text{etc.}$$

We find that

$$V(0,0) = \int_0^\infty \int_{-\infty}^m \frac{\lambda}{D} e^{-\alpha x-(\beta-\alpha)m}\pi(x,m)dxdm$$

Because this is for arbitrary π,

we conclude that ...

Another proof of 1):

- $$f_{X(\tau)}(x) \;=\; \kappa e^{-\alpha x}, \qquad \text{if} \;\; x < 0.$$

- Let $f_{X(\tau),\tau}(x,t)$ denote the joint probability density function of $X(\tau)$ and τ. Thus
$$f_{X(\tau)}(x) = \int_0^\infty e^{-\delta t} f_{X(\tau),\tau}(x,t)dt.$$

- Let $\widehat{f_{X(\tau)}}(z)$ denote the two-sided Laplace transform of $f_{X(\tau)}(x)$, we have
$$\begin{aligned}\widehat{f_{X(\tau)}}(z) &= \int_{-\infty}^\infty e^{-zx} f_{X(\tau)}(x)dx \\ &= \int_{-\infty}^\infty \int_0^\infty e^{-zx-\delta t} f_{X(\tau),\tau}(x,t)dtdx = \mathrm{E}[e^{-zX(\tau)-\delta\tau}].\end{aligned}$$

Another proof of 1):

- Let $f_\tau(t)$ denote the probability density function of τ and $\hat{f}(z)$ its Laplace transform. Then

$$\begin{aligned} \widehat{f_{X(\tau)}}(z) &= \mathrm{E}[\mathrm{E}[e^{-zX(\tau)-\delta\tau}|\tau]] = \mathrm{E}[e^{Dz^2\tau-\mu z\tau-\delta\tau}] \\ &= \hat{f}(-Dz^2+\mu z+\delta). \end{aligned}$$

- Note that $\widehat{f_{X(\tau)}}(z)$ is well defined for z such that $Dz^2-\mu z-\delta<0$; this is an open interval containing 0.

Another proof of 1):

- $$\hat{f}(z) = \frac{\lambda}{z + \lambda}$$

  and therefore,

  $$\widehat{f_{X(\tau)}}(z) = \frac{\lambda}{-Dz^2 + \mu z + \delta + \lambda}.$$

  We note that $-\beta$ and $-\alpha$ are the zeros of the denominator.
- $f_{X(\tau)}(x)$ can be obtained by inverting the Laplace transform.

Another proof of 1)

- For $m \geq \max(x, 0)$,

$$Pr(X(t) \leq x, M(t) > m) = e^{Rm} Pr(X(t) \leq x - 2m),$$

  where $R = \mu/D$ (the adjustment coefficient).
- Since this identity is true for each $t > 0$, we can replace t by T.
- For $x \leq 0$,

$$F_{X(\tau)}(x) = \frac{\kappa}{-\alpha} e^{-\alpha x} = \frac{\lambda}{-\alpha(\beta - \alpha)D} e^{-\alpha x}.$$

Another proof of 1)

- For $m \geq \max(x, 0)$,

$$\begin{aligned} & Pr(X(\tau) \leq x, M(\tau) > m) = e^{Rm} Pr(X(\tau) \leq x - 2m) \\ = \quad & e^{Rm} F_{X(\tau)}(x - 2m) = \frac{\lambda}{-\alpha(\beta - \alpha)D} e^{Rm - \alpha(x - 2m)} \\ = \quad & \frac{\lambda}{-\alpha(\beta - \alpha)D} e^{-(\beta - \alpha)m - \alpha x}. \end{aligned}$$

-

$$\begin{aligned} f_{X(\tau),M(\tau)}(x, m) &= -\frac{\partial^2}{\partial x \partial m} Pr(X(\tau) \leq x, M(\tau) > m) \\ &= \frac{\lambda}{D} e^{-(\beta - \alpha)m - \alpha x}, \qquad m \geq \max(x, 0). \end{aligned}$$

other ways to proof 1)

- The density $f_{X(t),M(t)}(x, m)$ is known
- $f^{\delta}_{X(\tau),M(\tau)}(x, m) = \int_0^{\infty} e^{-\delta t} f_{X(t),M(t)}(x, m) f_{\tau}(t) dt$
- $f_{X(\tau),M(\tau)}(x, m) = \int_0^{\infty} f_{X(t),M(t)}(x, m) f_{\tau}(t) dt$

Factorization formula

Lemma 1 : If τ is exponential with mean $1/\lambda$, then the following factorization formula holds,

$$E[e^{-\delta\tau} g_\tau(X)] = E[e^{-\delta\tau}] \times E[g_{\tau^*}(X)],$$

where τ^* is an exponential random variable with mean $1/(\lambda+\delta)$ and independent of X.
Remarks (i) $E[e^{-\delta\tau}] = \frac{\lambda}{\lambda+\delta}$.
(ii) The condition $\delta > 0$ can be replaced by the condition $\delta > -\lambda$.

Proof of the factorization formula

$$
\begin{aligned}
E[e^{-\delta\tau} g_\tau(X)] = \int_0^\infty e^{-\delta t} E[g_t(X)] \lambda e^{-\lambda t} dt \\
= \quad & \frac{\lambda}{\lambda+\delta} \int_0^\infty (\lambda+\delta) e^{-(\lambda+\delta)t} E[g_t(X)] dt \\
= \quad & E[e^{-\delta\tau}] \times E[g_{\tau^*}(X)].
\end{aligned}
$$

3. Financial applications

- $S(t)$: stock price
- $S(t) = S(0)e^{X(t)} = S(0)e^{\mu t + \sigma W(t)}, \quad t \geq 0$
- a contingent option provides a payoff at time τ
- Example: τ: time of death
  GMDB (Guaranteed Minimum Death Benefits)

A contingent option is exercised at time τ

Payoff:

- $[K - S(\tau)]_+$ contingent put option
- $[S(\tau) - K]_+$: contingent call option
- exotic expressions in terms of $S(\tau)$ and $\max_{0 \leq t \leq \tau} S(t)$
- $[K - S(\tau)]_+ I S_0 \, e^{M(\tau)} \geq H\}$ contingent up-and-in put option
  for $S(0) < H$ where H is the barrier level

The cost of the contingent put option

$$
\begin{aligned}
p &= \mathrm{E}[e^{-\delta\tau}[K - S(\tau)]_+] = \mathrm{E}[e^{-\delta\tau}[K - S(0)e^{X(\tau)}]_+] \\
&= \int_{-\infty}^{\ln(K/S(0))} [K - S(0)e^x] f^{\delta}_{X(\tau)}(x)dx, \\
&= \begin{cases} \frac{\kappa}{\alpha(\alpha-1)} K \left(\frac{K}{S(0)}\right)^{-\alpha} & \text{if } K \le S(0), \\ \frac{\kappa}{\beta(\beta-1)} K \left(\frac{K}{S(0)}\right)^{-\beta} + K\frac{\lambda}{\lambda+\delta} - S(0)\frac{\lambda}{\lambda+\delta-\mu-D} & \text{if } K > S(0). \end{cases}
\end{aligned}
$$

Other options

- barrier and double barrier options
- all or nothing options
- Margrabe option
- look back options
- policies has roll-up and/or dividends
- ...

T-year K-strike contingent put option

- Consider options that will expire at a fixed time T, $T > 0$. Thus, the time-τ payoff is

$$[K - S(\tau)]_+ I_{(\tau \le T)},$$

-

$$[K - S(\tau)]_+ - [K - S(\tau)]_+ I_{(\tau > T)}.$$

T-year K-strike contingent put option

$$
\begin{aligned}
& \mathrm{E}[e^{-\delta\tau}[K - S(\tau)]_+ I_{(\tau > T)}] \\
= \; & Pr(\tau > T)\mathrm{E}[e^{-\delta\tau}[K - S(\tau)]_+ | \tau > T] \\
= \; & e^{-(\lambda+\delta)T}\mathrm{E}[e^{-\delta\tau}[K - S(T)e^{\mu\tau + \sigma W(\tau)}]_+]
\end{aligned}
$$

by the memoryless property.

τ is an uniform distribution

- For τ exponential, define

$$V(\delta,\lambda,T) = E[e^{-\delta\tau}\pi(S(\tau),\tau)I_{\{\tau<T\}}] = \int_0^T \lambda e^{-(\lambda+\delta)t}E[\pi(S(t),t)]dt.$$

- For $\tau \sim U(0,T)$, the cost of option is

$$\int_0^T \frac{1}{T}e^{-\delta t}E[\pi(S(t),t)]dt = \frac{1}{T}\int_0^T e^{-\delta t}E[\pi(S(t),t)]dt$$
$$= \frac{1}{T}V(\delta,0,T).$$

Erlang distribution

- τ has an Erlang distribution

$$f_\tau(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, \qquad t > 0,$$

-

$$\hat{f}(z) = \left(\frac{\lambda}{z+\lambda} \right)^n.$$

Erlang distribution

- $$\begin{aligned} \widehat{f_{X(\tau)}}(z) &= \left(\frac{\lambda}{-Dz^2+\mu z+\delta+\lambda}\right)^n \\ &= \left(\frac{\lambda}{-D(z+\beta)(z+\alpha)}\right)^n \\ &= \kappa^n\left(\frac{1}{z+\beta}-\frac{1}{z+\alpha}\right)^n, \end{aligned}$$

- $$f_{X(\tau)}(x) = \begin{cases} \kappa^n e^{-\alpha x}\sum_{j=1}^n \frac{\binom{2n-j-1}{n-j}}{(j-1)!(\beta-\alpha)^{n-j}}(-x)^{j-1}, & \text{if} \quad x<0 \\ \kappa^n e^{-\beta x}\sum_{j=1}^n \frac{\binom{2n-j-1}{n-j}}{(j-1)!(\beta-\alpha)^{n-j}}x^{j-1}, & \text{if} \quad x>0 \end{cases}$$

A numerical example

- Let $\lambda = n/T$. Then $\mathrm{E}[\tau] = T$, $Var[\tau] = T^2/n$.
- $n \to \infty$, this family of Erlang distributions converges to the degenerate distribution at T.
- The price of a European T-year K-strike put option can be approximated by

$$p = \int_{-\infty}^{\ell} [K - S(0)e^x] f_{X(\tau)}(x) dx.$$

A numerical example

- Let $S(0) = 42,\ K = 40,\ \delta = 0.1,\ \sigma = 0.2$ and $T = 0.5$. Using the Black-Scholes formula, the put option price is 0.809.

Table: The prices of put option for various n

$n = 1$	$n = 10$	$n = 20$	$n = 30$	$n = 50$	$n = 100$	$n = 250$
0.624	0.786	0.797	0.801	0.804	0.806	0.808